

# $T^4$ fibrations over Calabi-Yau two-folds and non-Kähler manifolds in string theory

Hai Lin

*Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P. R. China*

## Abstract

We construct a geometric model of eight-dimensional manifolds and realize them in the context of type II string theory. These eight-manifolds are constructed by non-trivial  $T^4$  fibrations over Calabi-Yau two-folds. These give rise to eight-dimensional non-Kähler Hermitian manifolds with  $SU(4)$  structure. The eight-manifold is also a circle fibration over a seven-dimensional  $G_2$  manifold with skew torsion. The eight-manifolds of this type appear as internal manifolds with  $SU(4)$  structure in type IIB string theory with  $F_3$  and  $F_7$  fluxes. These manifolds have generalized calibrated cycles in the presence of fluxes.

# 1 Introduction

String theory has elegant and deep mathematical structures. It relates theoretical physics to mathematics and has provided great insights to both areas of research. In particular, a great number of important aspects of geometric questions have occurred and can be addressed in the context of string theory. For instance, manifolds with  $SU(n)$  structure, such as the Calabi-Yau  $n$ -folds, naturally appear in superstring theory and are important subjects for our understanding.

An interesting model of manifolds with  $SU(3)$  structure, is the geometric construction of  $T^2$  fibrations over Calabi-Yau two-folds [1, 2]. Such six-dimensional manifolds include not only Calabi-Yau three-folds of the Kähler type, but also non-Kähler Hermitian manifolds with  $SU(3)$  structure. They can appear as the internal six-manifolds when the superstring theory is compactified down to four-dimensional spacetime. A natural question that is addressed by this present paper is what happens if we use  $T^4$  fibrations, instead of  $T^2$  fibrations. This corresponds to a geometric model of eight-dimensional manifolds that we construct in this paper.

Internal manifolds with six dimensions have been well-studied, in the context of string compactification. However, eight-dimensional internal manifolds are also very interesting. They can have similar mathematical structures as their six-dimensional counterparts, for example they can be Hermitian and have an  $SU(n)$  structure where  $n$  is the complex dimension. Furthermore, balanced Hermitian manifolds exist in both six dimensions and eight dimensions. Moreover, eight-manifolds can naturally appear in the compactification of string theory with fluxes to two-dimensional spacetime.

Eight-dimensional manifolds with  $SU(4)$  structure include both Kähler Calabi-Yau four-folds and non-Kähler Hermitian manifolds with  $SU(4)$  structure. These manifolds are equipped with a Hermitian two-form and a holomorphic four-form. These forms can be constructed by bilinears of internal Killing spinors. These eight-dimensional manifolds have been studied by using the equations of pure spinors in type II string theory [3, 4, 5]. The Kähler Calabi-Yau four-folds are the special cases, when both the Hermitian form and holomorphic form are closed. In the presence of fluxes, these forms need not be closed, and this is the case for the non-Kähler  $SU(4)$ -structure manifolds.

The non-Kähler manifolds can appear naturally in string theory with fluxes. In the compactification of heterotic string theory to four dimensional Minkowski spacetime [6], the internal six-manifolds can become non-Kähler in the presence of fluxes [7, 8, 1, 9]. Various models of constructing heterotic manifolds and their vector-bundles have been put forward [7-13]. They play an important role in searching for realistic string theory vacua with four dimensional Minkowski spacetime.

An interesting type of non-Kähler manifolds, which are very important in differential geometry, are balanced Hermitian manifolds. They are Hermitian manifolds with a Hermitian form and a holomorphic form. For a balanced manifold, unlike Kähler manifolds, its Hermitian form is not closed, however, the  $(n - 1)$ th power of its Hermitian

Hermitian form is closed, where  $n$  is the complex dimension of the manifold [14]. Since they impose a weaker condition on the closure of the Hermitian form than the Kähler manifolds, they represent close variants of Kähler manifolds. Some non-Kähler Hermitian balanced manifolds can have trivial canonical bundle, and thus are interesting examples of non-Kähler Calabi-Yau manifolds, see for instance [15]. Moreover, under appropriate blowing-downs or contractions of curves, some classes of balanced manifolds can become Kähler and have projective models in algebraic geometry.

In this paper, we will construct eight-dimensional manifolds of the non-Kähler Hermitian type, by  $T^4$  fibrations over Calabi-Yau two-folds. They have  $SU(4)$  structures but are not the standard Kähler Calabi-Yau four-folds. The eight-manifolds can also be viewed as a circle bundle over a seven-dimensional base. We will show that the base is a  $G_2$  manifold with skew torsion. General  $G_2$  manifolds with torsion have been widely studied [16-20]. The geometric model of the eight-manifolds here, fits with type II string theory with  $F_3$  and  $F_7$  fluxes and dilaton, as we will see in the later sections.

The organization of this paper is as follows. In Sec. 2, we construct eight dimensional Hermitian manifolds by  $T^4$  fibrations over Calabi-Yau two-folds. In Sec. 3, we find that the eight-manifold of this type can be viewed as a circle bundle over a seven-dimensional  $G_2$  manifold with skew torsion. After that in Sec. 4, we find that the eight-manifold of this kind can be used in type IIB string theory on the warped product of a two-dimensional Minkowski spacetime and an eight-manifold. Then in Sec. 5, generalized calibration forms and generalized calibrated cycles are constructed for these models appearing in the type IIB string theory. Finally we briefly discuss related aspects in Sec. 6.

## 2 $T^4$ fibrations over Calabi-Yau two-folds and non-Kähler eight-manifolds

In this section we construct a geometric model of eight-dimensional Hermitian manifolds, by fibrations of four-dimensional tori  $T^4$  over four-dimensional base manifolds which are complex. We devote particular attention to the case that the four dimensional base is a Calabi-Yau two-fold.

Let us consider a ten-dimensional metric of string theory arising as a warped product of a two-dimensional Minkowski spacetime  $R^{1,1}$  and an eight-dimensional manifold  $M^8$ . The line element of the ten-dimensional metric is

$$ds^2 = e^{2A} ds^2(R^{1,1}) + ds^2(M^8). \quad (1)$$

Here,  $M^8$  is a non-trivial  $T^4$  fibration over a four-manifold  $M^4$

$$T^4 \rightarrow M^8 \rightarrow M^4. \quad (2)$$

We define the projection map

$$\pi : M^8 \rightarrow M^4. \quad (3)$$

In general, we can consider the eight-dimensional manifold  $M^8$  to be either compact or non-compact. For instance, we can obtain non-compact  $M^8$  by taking the base  $M^4$  to be non-compact. The  $e^{2A}$  in the metric (1) is a warp factor in front of the metric of  $R^{1,1}$ .

The line element of the eight-dimensional metric is

$$ds^2(M^8) = e^{2v}[\text{Re}(\theta_{(1)} \otimes \bar{\theta}_{(1)} + \theta_{(2)} \otimes \bar{\theta}_{(2)}) + e^{2C} ds^2(M^4)], \quad (4)$$

where

$$\theta_{(1)} = dx_1 + idy_1 + A_{(1)}, \quad (5)$$

$$\theta_{(2)} = dx_2 + idy_2 + A_{(2)}. \quad (6)$$

We consider  $M^4$  as a complex manifold, equipped with a Hermitian two-form  $J_{M^4}$  and a holomorphic two-form  $\Omega_{M^4}$ , so that  $d\Omega_{M^4} = 0$ . The  $\{x_1, y_1, x_2, y_2\}$  are coordinates of the tori  $T^4$ . The connections of the fibrations are complex one-forms  $A_{(1)}$  and  $A_{(2)}$ . Their curvatures are  $F_{(i)} = dA_{(i)}$  and  $\bar{F}_{(i)} = d\bar{A}_{(i)}$ , for  $i = 1, 2$ . The  $e^{2A}, e^{2v}, e^{2v+2C}$  are three warped factors. The  $A, v, C$  are functions on the four-manifold  $M^4$ . The function  $e^{2C}$  is a warp factor in front of the metric of  $M^4$ .

The line element of the  $M^4$  can be written as

$$ds^2(\tilde{M}^4) = e^{2C} ds^2(M^4), \quad (7)$$

where  $\tilde{M}^4$  is a Hermitian manifold with  $J_{\tilde{M}^4} = e^{2C} J_{M^4}$  and  $\Omega_{\tilde{M}^4} = e^{2C} \Omega_{M^4}$ .

Now let us describe the geometry of the  $T^4$  fibration in more detail. The  $A_{(1)}$  and  $A_{(2)}$  are the pull-backs of the complex one-forms  $a_{(1)}$  and  $a_{(2)}$  on the base complex four-manifold  $M^4$ . In other words,  $A_{(1)} = \pi^* a_{(1)}$  and  $A_{(2)} = \pi^* a_{(2)}$ . We assume that  $a_{(1)}$  and  $a_{(2)}$  are of the  $(1, 0)$  type on the base. In component form, the fibration of the  $T^4$  is described by

$$(dx_1 + \pi^* \text{Re } a_{(1)})^2 + (dy_1 + \pi^* \text{Im } a_{(1)})^2 + (dx_2 + \pi^* \text{Re } a_{(2)})^2 + (dy_2 + \pi^* \text{Im } a_{(2)})^2. \quad (8)$$

The curvatures  $f_{(1)}$  and  $f_{(2)}$  of the complex one-forms on the base can be written locally as  $f_{(1)} = da_{(1)}$  and  $f_{(2)} = da_{(2)}$ . The connections  $a_{(i)}$  and  $\bar{a}_{(i)}$ , for  $i = 1, 2$ , have curvatures such that  $[-\frac{f_{(i)}}{2\pi}], [-\frac{\bar{f}_{(i)}}{2\pi}] \in H^2(M^4, \mathbb{Z})$ . We see that  $F_{(i)} = \pi^* f_{(i)}$  and  $\bar{F}_{(i)} = \pi^* \bar{f}_{(i)}$ .

The eight-manifold  $M^8$  is hence a Hermitian manifold equipped with a Riemannian metric in (4), a Hermitian  $(1, 1)$  form  $J$ , and a holomorphic  $(4, 0)$  form  $\Omega$ . Let us denote

$$J_{(1)} = \frac{i}{2} \theta_{(1)} \wedge \bar{\theta}_{(1)}, \quad J_{(2)} = \frac{i}{2} \theta_{(2)} \wedge \bar{\theta}_{(2)}. \quad (9)$$

The Hermitian form  $J$  and the holomorphic form  $\Omega$  of  $M^8$  are

$$J = e^{2v} J_{(1)} + e^{2v} J_{(2)} + e^{2(v+C)} \pi^* J_{M^4}, \quad (10)$$

$$\Omega = e^{4v+2C} \theta_{(1)} \wedge \theta_{(2)} \wedge \pi^* \Omega_{M^4}. \quad (11)$$

We have that  $J_{(1)}^2 = 0$ ,  $J_{(2)}^2 = 0$ ,  $\pi^* J_{M^4}^3 = 0$ . The holomorphic  $(4, 0)$  form  $\Omega$  requires that  $\theta_{(1)}$  and  $\theta_{(2)}$ , and hence  $A_{(1)}$  and  $A_{(2)}$ , are of the  $(1, 0)$  type. This is the reason that we have assumed that  $a_{(1)}$  and  $a_{(2)}$  are of the  $(1, 0)$  type on the base.

Let us consider the closure properties of  $\Omega$  and  $J$ . We first analyze  $\Omega$ ,

$$d(e^{-4v-2C} \Omega) = (F_{(1)} \wedge \theta_{(2)} - F_{(2)} \wedge \theta_{(1)}) \wedge \pi^* \Omega_{M^4}. \quad (12)$$

By demanding the vanishing of the right-hand side of Eq. (12), we assume the condition

$$F_{(i)} \wedge \pi^* \Omega_{M^4} = 0. \quad (13)$$

Hence, with this condition

$$d(e^{-4v-2C} \Omega) = 0. \quad (14)$$

Now let us consider the closure property of  $J$ ,

$$\begin{aligned} dJ &= e^{2v} \left( \frac{i}{2} F_{(1)} \wedge \bar{\theta}_{(1)} - \frac{i}{2} \bar{F}_{(1)} \wedge \theta_{(1)} + \frac{i}{2} F_{(2)} \wedge \bar{\theta}_{(2)} - \frac{i}{2} \bar{F}_{(2)} \wedge \theta_{(2)} \right) \\ &\quad + 2dv \wedge J + 2e^{2v+2C} dC \wedge \pi^* J_{M^4} + e^{2v+2C} \pi^* dJ_{M^4}. \end{aligned} \quad (15)$$

Due to the presence of nonzero  $F_{(1)}, \bar{F}_{(1)}, F_{(2)}, \bar{F}_{(2)}$ , the first line can not vanish. In other words,

$$\begin{aligned} &\frac{i}{2} F_{(1)} \wedge \bar{\theta}_{(1)} - \frac{i}{2} \bar{F}_{(1)} \wedge \theta_{(1)} + \frac{i}{2} F_{(2)} \wedge \bar{\theta}_{(2)} - \frac{i}{2} \bar{F}_{(2)} \wedge \theta_{(2)} \\ &= -\text{Im}(F_{(1)} \wedge \bar{\theta}_{(1)} + F_{(2)} \wedge \bar{\theta}_{(2)}) \neq 0. \end{aligned} \quad (16)$$

Therefore  $J$  is not closed or conformally closed. Hence, with the nonzero  $F_{(i)}, \bar{F}_{(i)}$ , the eight-manifold  $M^8$  is not Kähler and not conformally Kähler.

If  $M^4$  is complex and non-Kähler, then the non-Kählerity of  $M^8$  can be attributed to the base  $M^4$  being non-Kähler, as from the last term in (15). To analyze situations when the non-Kählerity of  $M^8$  is not attributed to the base  $M^4$  being non-Kähler, we consider  $M^4$  being Kähler, in other words,

$$dJ_{M^4} = 0. \quad (17)$$

$\tilde{M}^4$  is hence conformally Kähler.

Let us now consider

$$\begin{aligned} d(J^2) &= -\frac{1}{2} e^{4v} (F_{(1)} \wedge \theta_{(2)} \wedge \bar{\theta}_{(2)} \wedge \bar{\theta}_{(1)} - \bar{F}_{(1)} \wedge \theta_{(2)} \wedge \bar{\theta}_{(2)} \wedge \theta_{(1)} \\ &\quad + F_{(2)} \wedge \theta_{(1)} \wedge \bar{\theta}_{(1)} \wedge \bar{\theta}_{(2)} - \bar{F}_{(2)} \wedge \theta_{(1)} \wedge \bar{\theta}_{(1)} \wedge \theta_{(2)}) \\ &\quad + 4dv \wedge J^2 + 4e^{4v+2C} dC \wedge \pi^* J_{M^4} \wedge (J_{(1)} + J_{(2)}), \end{aligned} \quad (18)$$

where we have used  $dJ_{M^4} = 0$  and assumed the condition

$$F_{(i)} \wedge \pi^* J_{M^4} = 0, \quad \bar{F}_{(i)} \wedge \pi^* J_{M^4} = 0. \quad (19)$$

Due to the nonzero  $F_{(1)}, \bar{F}_{(1)}, F_{(2)}, \bar{F}_{(2)}$ , the  $J^2$  is not closed or conformally closed.

Finally let us consider  $dJ^3$ . With the conditions (17) and (19),

$$d(e^{-6v-2C} J^3) = 0. \quad (20)$$

Hence in this case  $J^3$  is a conformally closed  $(3, 3)$  form.

The eight-manifold  $M^8$  can be written as

$$ds^2(M^8) = e^{2v} ds^2(\tilde{M}^8), \quad (21)$$

$$ds^2(\tilde{M}^8) = \text{Re}(\theta_{(1)} \otimes \bar{\theta}_{(1)} + \theta_{(2)} \otimes \bar{\theta}_{(2)}) + e^{2C} ds^2(M^4), \quad (22)$$

where  $M^8$  is conformal to  $\tilde{M}^8$ . The  $\tilde{M}^8$  has a Hermitian two-form  $\tilde{J}$  and a holomorphic four-form  $\tilde{\Omega}$  as follows,

$$\tilde{J} = J_{(1)} + J_{(2)} + e^{2C} \pi^* J_{M^4}, \quad (23)$$

$$\tilde{\Omega} = \theta_{(1)} \wedge \theta_{(2)} \wedge \pi^* \Omega_{M^4}. \quad (24)$$

The norm of  $\tilde{\Omega}$  with respect to the Hermitian form  $\tilde{J}$  is

$$\|\tilde{\Omega}\|_{\tilde{J}} = e^{-2C}. \quad (25)$$

We have that  $d(e^{-2C} \tilde{J}^3) = 0$ , and from Eq. (25), we see that

$$d(\|\tilde{\Omega}\|_{\tilde{J}} \tilde{J}^3) = 0. \quad (26)$$

This expression (26) is for the ansatz in Eqs. (23) and (24). We have assumed that  $M^4$  is Kähler in the above derivation of Eq. (26). In order that the holomorphic four-form  $\tilde{\Omega}$  is non-vanishing, according to Eq. (24),  $M^4$  has a non-vanishing holomorphic two-form. Hence, by the classification of complex surfaces by Enriques and Kodaira,  $M^4$  are Calabi-Yau two-folds. Under a conformal transformation, let  $\tilde{J}' = e^{-\frac{2C}{3}} \tilde{J}$ , then  $d\tilde{J}'^3 = 0$ . This is the condition for eight-dimensional conformally balanced manifolds [14]. Hence,  $M^8$  is conformally balanced, with the additional assumption used in the above derivation

$$\begin{aligned} F_{(i)} \wedge \pi^* \Omega_{M^4} &= 0, \\ F_{(i)} \wedge \pi^* J_{M^4} &= 0, \\ \bar{F}_{(i)} \wedge \pi^* J_{M^4} &= 0. \end{aligned} \quad (27)$$

The balanced manifolds have certain nice properties. Some balanced manifolds, although not Kähler, after performing appropriate blowing-downs or contractions of curves, have a limit that become projective and Kähler, see for example [21, 22, 23,

24, 25]. Some smooth balanced manifolds can appear as crepant resolutions of certain projective and Kähler manifolds, see for example the six dimensional case discussed in [21, 22, 23, 24].

Now let us consider what the condition (27) imply for the base  $M^4$  and the fibrations of  $T^4$ . The space of the two-forms on  $M^4$  can be decomposed by the direct sum of the space of self-dual two-forms  $\Omega_{M^4}^{2+}$  and the space of anti-self-dual two-forms  $\Omega_{M^4}^{2-}$ ,

$$\Omega_{M^4}^2 = \Omega_{M^4}^{2+} \oplus \Omega_{M^4}^{2-}. \quad (28)$$

For a complex manifold  $M^4$ , it can be further decomposed as

$$\Omega_{M^4}^{2+} = \Omega_{M^4}^{2,0} \oplus \Omega_{M^4}^{1,1+} \oplus \Omega_{M^4}^{0,2}, \quad \Omega_{M^4}^{2-} = \Omega_{M^4}^{1,1-}, \quad (29)$$

where the superscripts  $+$  and  $-$  mean self-dual and anti-self-dual, respectively. The two-forms  $J_{M^4}$ ,  $\Omega_{M^4}$ , and  $\bar{\Omega}_{M^4}$  are in the spaces  $\Omega_{M^4}^{1,1+}$ ,  $\Omega_{M^4}^{2,0}$  and  $\Omega_{M^4}^{0,2}$  respectively. The condition (27) implies that

$$\begin{aligned} f_{(i)} \wedge J_{M^4} &= 0, & \bar{f}_{(i)} \wedge J_{M^4} &= 0, \\ f_{(i)} \wedge \Omega_{M^4} &= 0, & \bar{f}_{(i)} \wedge \Omega_{M^4} &= 0. \end{aligned} \quad (30)$$

This means that  $f_{(i)}$ ,  $\bar{f}_{(i)}$  are perpendicular to  $J_{M^4}$ ,  $\Omega_{M^4}$ ,  $\bar{\Omega}_{M^4}$ , and are thus in the space  $\Omega_{M^4}^{1,1-}$ . The first three equations imply the fourth equation in (30), by the decomposition in (29). Since  $f_{(i)}, \bar{f}_{(i)} \in \Omega_{M^4}^{1,1-}$ , the connections  $a_{(i)}, \bar{a}_{(i)}$  have anti-self-dual curvatures, that is,

$$\begin{aligned} f_{(i)} &= - * f_{(i)}, \\ \bar{f}_{(i)} &= - * \bar{f}_{(i)}. \end{aligned} \quad (31)$$

Connections with anti-self-dual curvatures on four-manifolds have been discussed in, for example [26, 27]. The  $f_{(i)}, \bar{f}_{(i)}$  are of  $(1, 1)$  type here. Moreover, they are orthogonal to the self-dual two forms. They are primitive  $(1, 1)$  forms. We hence refer to Eqs. (30) and (27) as primitivity condition.

The metric ansatz (4) of  $M^8$  fits into special cases of eight-manifolds considered in [3, 4, 28, 29]. Let us consider the condition of  $SU(4)$  structure for  $M^8$ . The  $SU(4)$  structure relation is given by

$$\frac{1}{2^4} \Omega \wedge \bar{\Omega} = \frac{1}{4!} J^4, \quad J \wedge \Omega = 0. \quad (32)$$

From the above ansatz (4), (10) and (11),

$$\frac{1}{2^4} \Omega \wedge \bar{\Omega} = \frac{1}{4} J_{(1)} \wedge J_{(2)} \wedge \pi^*(\Omega_{M^4} \wedge \bar{\Omega}_{M^4}) e^{8v+4C}, \quad (33)$$

$$\frac{1}{4!} J^4 = \frac{1}{2} J_{(1)} \wedge J_{(2)} \wedge \pi^* J_{M^4}^2 e^{8v+4C}. \quad (34)$$

The  $SU(4)$  condition

$$\frac{1}{2^4}\Omega \wedge \bar{\Omega} = \frac{1}{4!}J^4 \quad (35)$$

requires that

$$\frac{1}{2^2}\Omega_{M^4} \wedge \bar{\Omega}_{M^4} = \frac{1}{2!}J_{M^4}^2, \quad (36)$$

and the  $SU(4)$  condition

$$J \wedge \Omega = e^{6v+4C}\theta_{(1)} \wedge \theta_{(2)} \wedge \pi^*(J_{M^4} \wedge \Omega_{M^4}) = 0 \quad (37)$$

requires that

$$J_{M^4} \wedge \Omega_{M^4} = 0. \quad (38)$$

The Eqs. (36) and (38) mean that the base  $M^4$  has  $SU(2)$  structure. Since  $M^4$  is also Kähler, this means that  $M^4$  is a Calabi-Yau two-fold. The general Calabi-Yau two-folds include both compact Calabi-Yau two-folds such as K3 surfaces and non-compact Calabi-Yau two-folds. In order that the model of  $M^8$  has  $SU(4)$  structure, the base complex manifold  $M^4$  is a Calabi-Yau two-fold. Moreover, we have also showed that when the base  $M^4$  is a Calabi-Yau two-fold, with the additional assumption of the primitivity condition (30), the  $M^8$  is a conformally balanced  $SU(4)$ -structure Hermitian manifold.

### 3 $G_2$ manifolds from the eight-manifolds

In the previous section we have constructed a geometric model of eight-dimensional manifolds by considering the manifold  $M^8$  as a  $T^4$  fibration over of a  $M^4$  base. In this section we describe the  $M^8$  in another way. The  $M^8$  can be viewed as a circle fibration over a seven-dimensional manifold  $M^7$ ,

$$S^1 \rightarrow M^8 \rightarrow M^7. \quad (39)$$

We define the map

$$\tau : M^8 \rightarrow M^7. \quad (40)$$

According to the metric ansatz (4), the  $M^7$  is hence the  $T^3$  fibration over  $M^4$ ,

$$T^3 \rightarrow M^7 \rightarrow M^4 \quad (41)$$

and we define the map

$$\psi : M^7 \rightarrow M^4. \quad (42)$$

The projection map  $\pi$  in Sec. 2 is hence

$$\pi = \psi \circ \tau. \quad (43)$$



As in Sec. 2, we consider the base  $M^4$  to be a Calabi-Yau two-fold, which has  $SU(2)$  structure. We will see in this section that  $M^7$  is a  $G_2$  manifold with skew torsion.

A  $G_2$  manifold with torsion contains a metric, a fundamental three-form  $\varphi_3$ , and its dual four-form  $\varphi_4 = *\varphi_3$ . If it has torsion, then  $d\varphi_3 \neq 0$ , and the  $d\varphi_3$  measures the torsion. For the classifications of  $G_2$  manifolds with torsion, see for example [16, 17, 18, 19, 20].

The metric ansatz of the eight-manifold is

$$ds^2(M^8) = e^{2v}(dx_1 + \tau^*\psi^* \operatorname{Re} a_{(1)})^2 + ds^2(\tilde{M}^7) \quad (44)$$

where the  $x_1$  parametrizes the coordinate of the  $S^1$  and

$$ds^2(\tilde{M}^7) = e^{2v}ds^2(M^7). \quad (45)$$

The seven-manifold we are looking at is

$$ds^2(M^7) = (dy_1 + \psi^* \operatorname{Im} a_{(1)})^2 + (dx_2 + \psi^* \operatorname{Re} a_{(2)})^2 + (dy_2 + \psi^* \operatorname{Im} a_{(2)})^2 + e^{2C}ds^2(M^4), \quad (46)$$

where the  $\{y_1, x_2, y_2\}$  are coordinates of the  $T^3$ .

We can define the fundamental three-form  $\varphi_3$  of  $M^7$ ,

$$\begin{aligned} \varphi_3 &= (J_{(2)} + e^{2C}\psi^*J_{M^4}) \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}) + e^{2C} \operatorname{Im}(\theta_{(2)} \wedge \psi^*\Omega_{M^4}) \\ &= (dy_1 + \psi^* \operatorname{Im} a_{(1)}) \wedge (dx_2 + \psi^* \operatorname{Re} a_{(2)}) \wedge (dy_2 + \psi^* \operatorname{Im} a_{(2)}) \\ &\quad + e^{2C}[\psi^*J_{M^4} \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}) + \operatorname{Im}(\theta_{(2)} \wedge \psi^*\Omega_{M^4})]. \end{aligned} \quad (47)$$

The dual four-form  $\varphi_4$  is

$$\begin{aligned} \varphi_4 &= *\varphi_3 \\ &= \frac{1}{2}(J_{(2)} + e^{2C}\psi^*J_{M^4})^2 - e^{2C} \operatorname{Re}(\theta_{(2)} \wedge \psi^*\Omega_{M^4}) \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}). \end{aligned} \quad (48)$$

We see that

$$\begin{aligned} d\varphi_3 &= \psi^* \operatorname{Im} f_{(2)} \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}) \wedge (dx_2 + \psi^* \operatorname{Re} a_{(2)}) \\ &\quad + \psi^* \operatorname{Re} f_{(2)} \wedge (dy_2 + \psi^* \operatorname{Im} a_{(2)}) \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}) \\ &\quad + \psi^* \operatorname{Im} f_{(1)} \wedge (dx_2 + \psi^* \operatorname{Re} a_{(2)}) \wedge (dy_2 + \psi^* \operatorname{Im} a_{(2)}) \\ &\quad + 2e^{2C}dC \wedge [\psi^*J_{M^4} \wedge (dy_1 + \psi^* \operatorname{Im} a_{(1)}) + \operatorname{Im}(\theta_{(2)} \wedge \psi^*\Omega_{M^4})]. \end{aligned} \quad (49)$$

Due to the non-zero  $\psi^* \operatorname{Im} f_{(2)}, \psi^* \operatorname{Re} f_{(2)}, \psi^* \operatorname{Im} f_{(1)}$  in Eq. (49),  $\varphi_3$  is not closed, even if after a rescaling. In Eq. (49), the primitivity condition (30) has been used. We also see that

$$\varphi_3 \wedge d\varphi_3 = 0, \quad (50)$$

in which we have used the primitivity condition and the  $SU(2)$  structure relation  $J_{M^4} \wedge \Omega_{M^4} = 0$ . Meanwhile,

$$\begin{aligned} d\varphi_4 &= 2e^{2C} dC \wedge [J_{(2)} \wedge \psi^* J_{M^4} - \text{Re}(\theta_{(2)} \wedge \psi^* \Omega_{M^4}) \wedge (dy_1 + \psi^* \text{Im } a_{(1)})] \\ &= 2e^{2C} dC \wedge \varphi_4 \\ &= \hat{\theta} \wedge \varphi_4, \end{aligned} \tag{51}$$

where  $\hat{\theta}$  is a Lee one-form

$$\hat{\theta} = 2e^{2C} dC. \tag{52}$$

Since

$$d\varphi_4 = \hat{\theta} \wedge \varphi_4, \quad d\varphi_3 \neq 0, \quad \varphi_3 \wedge d\varphi_3 = 0, \tag{53}$$

this is a  $G_2$  structure with skew torsion [16, 17, 18, 19, 20]. Since the Lee one-form  $\hat{\theta}$  is closed, that is  $d\hat{\theta} = 0$ , this  $G_2$  structure is locally conformal to a balanced  $G_2$  structure, see [16, 17, 18, 19, 20]. Hence, we see that  $(M^7, g_{M^7}, \varphi_3)$  constructed from the  $T^3$  fibration over a Calabi-Yau two-fold gives a  $G_2$  manifold with skew torsion.

In the special case if  $\hat{\theta} = 2e^{2C} dC$  vanish, then

$$d\varphi_4 = 0, \quad d\varphi_3 \neq 0, \quad \varphi_3 \wedge d\varphi_3 = 0. \tag{54}$$

This is the condition of a balanced  $G_2$  manifold [16, 17, 18, 19, 20, 30]. Hence, in this case, the seven-manifold  $M^7$  is a balanced  $G_2$  manifold.

In the above derivation, we have showed that  $M^8$  is a circle fibration over a  $G_2$  manifold with skew torsion. The seven-manifold  $M^7$ , as a  $T^3$  bundle over  $M^4$ , can be described in another way. Let us define the projection maps

$$\varrho : M^7 \rightarrow M^6, \tag{55}$$

$$\varsigma : M^6 \rightarrow M^4, \tag{56}$$

where  $M^6$  is described by

$$ds^2(M^6) = (dx_2 + \varsigma^* \text{Re } a_{(2)})^2 + (dy_2 + \varsigma^* \text{Im } a_{(2)})^2 + e^{2C} ds^2(M^4), \tag{57}$$

$$J_{M^6} = J_{(2)} + e^{2C} \varsigma^* J_{M^4}, \quad \Omega_{M^6} = e^{2C} \theta_{(2)} \wedge \varsigma^* \Omega_{M^4}. \tag{58}$$

The  $M^6$  is a conformally balanced Hermitian manifold. Hence, the projection map (42) can be written as

$$\psi = \varsigma \circ \varrho. \tag{59}$$

Hence  $M^7$  is also a circle fibration over  $M^6$ . This circle is parametrized by  $y_1$ . Considering it as a circle fibration of  $M^6$ , according to [31], we may also see that the Eqs. (47) and (48) can also be written as

$$\varphi_3 = \varrho^* J_{M^6} \wedge (dy_1 + \varrho^* \varsigma^* \text{Im } a_{(1)}) + \varrho^* \text{Im}(\Omega_{M^6}), \tag{60}$$

$$\varphi_4 = \frac{1}{2} \varrho^* J_{M^6} \wedge \varrho^* J_{M^6} - \varrho^* \text{Re}(\Omega_{M^6}) \wedge (dy_1 + \varrho^* \varsigma^* \text{Im } a_{(1)}). \tag{61}$$

By an analysis similar to the one in Sec. 2, the base  $M^4$  satisfies the  $SU(2)$  structure relation (36) and (38). Hence the base  $M^4$  has  $SU(2)$  structure and we have showed in the above that the  $T^3$  bundle over  $M^4$  has  $G_2$  structure with skew torsion.

## 4 $SU(4)$ structures and fluxes

The previous sections have described the construction of the eight-dimensional manifolds and their geometric properties. Let us now discuss how these eight-manifolds can be used in string theory. Let us consider to embed the metric ansatz (4) of Sec. 2 in type II string theory. The ten-dimensional spacetime is a warped product of a two-dimensional Minkowski spacetime and an eight-dimensional manifold  $M^8$ , with the line element

$$ds^2 = e^{2A} ds^2(R^{1,1}) + ds^2(M^8), \quad (62)$$

where  $e^{2A}$  is a warp factor in front of the metric of  $R^{1,1}$ . There is also a dilaton field  $\phi$  in the ten-dimensional spacetime. The Poincaré invariance in two-dimensional Minkowski spacetime and the self-duality constraint of the fluxes enables the decomposition [3, 4, 5] of the fluxes as

$$\mathcal{F} = \text{Vol}_2 \wedge e^{2A} *_8 \sigma F + F. \quad (63)$$

Here,  $\text{Vol}_2$  is the volume form of  $R^{1,1}$ . The  $\mathcal{F}$  is a polyform, which is the sum of the R-R fluxes of different ranks. The  $F$  in the ansatz (63) is a polyform on the internal manifold. Let us restrict our attention to type IIB, in which case  $\mathcal{F} = \sum \mathcal{F}_{(k)}$ , where  $k = 1, 3, 5, 7, 9$ . The  $\sigma$  is a sign factor, and  $\sigma \mathcal{F}_{(k)} = (-1)^{\frac{1}{2}k(k-1)} \mathcal{F}_{(k)}$  where  $k$  is the rank of the form. The self-duality constraint in type IIB theory is

$$\mathcal{F} = *_{10} \sigma \mathcal{F}, \quad (64)$$

and is satisfied by the ansatz (63).

The type IIB string theory in ten dimensions has two Killing spinors  $\epsilon_1, \epsilon_2$  of the same chirality. This case corresponds to  $\mathcal{N} = (2, 0)$  supersymmetry in 1+1 dimensions. There are two positive chirality supercharges, which can be denoted by a complex-valued Weyl spinor  $\zeta$  in 1+1 dimensions. For these solutions the most general decomposition of the Killing spinors  $\epsilon_1, \epsilon_2$  is given by

$$\epsilon_1 = \zeta \otimes \eta_1 + \text{c.c.} \quad (65)$$

$$\epsilon_2 = \zeta \otimes \eta_2 + \text{c.c.} \quad (66)$$

where  $\eta_1, \eta_2$  are internal Killing spinors which are Weyl spinors in 8 dimensions and they have the same chirality. The  $M_8$  is equipped with an  $SU(4)$  structure, which is equivalent to the existence of a pure spinor  $\eta$ . In this case, the pure spinor is  $\eta \propto \eta_1 = e^{-i\vartheta} \eta_2$ , up to a normalization factor. The pure spinor  $\eta$  satisfies  $\eta^t \eta = 0$ . One

can construct an  $SU(4)$  structure by taking the spinor bilinears [3, 4] of the internal Killing spinor

$$J_{mn} = -i\eta^\dagger \gamma_{mn} \eta, \quad (67)$$

$$\Omega_{mnpq} = \eta^t \gamma_{mnpq} \eta. \quad (68)$$

By using Fierz identities one can show that these forms obey the  $SU(4)$  structure relation (32). The  $J$  and  $\Omega$  are the Hermitian two-form and holomorphic four-form respectively. The polyforms can be written as

$$\Psi_1 = -e^{-i\vartheta} e^{-iJ}, \quad (69)$$

$$\Psi_2 = -e^{i\vartheta} \Omega, \quad (70)$$

where  $e^{-i\vartheta}$  is a phase factor. For more discussions on the properties of pure spinors, see for example [3, 4, 32, 33, 34, 35, 36].

It can be shown [3, 4] that the supersymmetry equations can be elegantly written with the pure spinors as

$$d_H (e^{2A-\phi} \text{Re}\Psi_1) = e^{2A} * \sigma F, \quad (71)$$

$$d_H (e^{2A-\phi} \Psi_2) = 0, \quad (72)$$

$$i(\bar{\partial}_H - \partial_H) (e^{-\phi} \text{Im}\Psi_1) = F, \quad (73)$$

where  $d_H$  is the twisted exterior derivative, and  $d_H \equiv d + H \wedge$ , where  $H$  is the NS-NS three form. We can decompose it as  $d_H = \partial_H + \bar{\partial}_H$ , where  $\partial_H \equiv \partial + H^{(2,1)} \wedge$  is the ordinary twisted Dolbeault operator and  $\bar{\partial}_H = \bar{\partial} + H^{(1,2)} \wedge$  is its complex conjugate, and  $H^{(2,1)}$ ,  $H^{(1,2)}$  are the  $(2,1)$  type and  $(1,2)$  type in  $H$ . More details on these equations and their generalizations have been discussed in [3, 4].

In the absence of  $H$ ,  $d_H$  reduces to  $d$ , and  $i(\bar{\partial}_H - \partial_H)$  reduces to  $i(\bar{\partial} - \partial) = d^c$ . Let us consider also the absence of  $F_1$  and  $F_5$ . From the differential equations for pure spinors (71) and (73), we have respectively

$$*_8 F_3 = \frac{1}{2} e^{-2A} d(e^{2A-\phi} J^2), \quad (74)$$

and

$$F_3 = i(\partial - \bar{\partial})(e^{-\phi} J) = -d^c(e^{-\phi} J). \quad (75)$$

Let us now combine the above two equations (74) and (75), and then we have

$$d^c(e^{-\phi} J) = \frac{1}{2} e^{-2A} *_8 d(e^{2A-\phi} J^2). \quad (76)$$

The case with constant  $e^\phi$  and  $e^{2A}$  is  $d^c J = \frac{1}{2} *_8 d(J^2)$ , which was obtained in [29]. Since  $d(e^{2A-\phi} J^2)$  appears in  $F_7$ , and  $d^c(e^{-\phi} J)$  appears in  $F_3$ , the Hodge dual relation (76) is closely connected to the self-duality constraint (64) in the type IIB string theory.

The IIB theory contains the field equations [37, 38] in the string frame

$$d * (\tilde{F}_3) = g_s F_5 \wedge H_3, \quad (77)$$

where  $\tilde{F}_3 = F_3 + C_0 H_3$ ,  $F_5 = dC_4$ , and  $C_0$  is the axion. In the case without the axion and  $F_5$ , this equation reduces to  $dF_7 = 0$  in our convention. This equation is equivalent to the Bianchi identity from the pure spinor equations.

The Eq. (72) gives

$$d(e^{2A-\phi}\Omega) = 0. \quad (78)$$

We use the ansatz (4) for the eight-manifolds  $M^8$  with  $SU(4)$  structure which appear in the warped product (62), in the case of type IIB string theory. Comparing Eq. (14) with Eq. (78), we see that

$$e^{2C} = e^{\phi-4v-2A}. \quad (79)$$

In general, we can consider the eight-manifolds  $M^8$  to be either compact or non-compact. For instance, we can obtain non-compact  $M^8$  by taking the base  $M^4$  in the ansatz (4) to be non-compact. The eight-dimensional non-compact models in the ten-dimensional string theory may be considered as local models of compact solutions.

If demanding  $d\Omega = 0$ , which means that  $M^8$  has an integrable complex structure, we have

$$e^\phi = e^{2A}. \quad (80)$$

However, this equation (80) should correspond to a special case, which should lead to special solutions.

The  $F_3$  is

$$F_3 = -d^c(e^{-\phi}J) = i(\partial - \bar{\partial})(e^{2v-\phi}(J_{(1)} + J_{(2)} + e^{2C}\pi^*J_{M^4})) \quad (81)$$

$$= e^{2v-\phi}(\text{Re}(\bar{F}_{(1)} \wedge \theta_{(1)}) + \text{Re}(\bar{F}_{(2)} \wedge \theta_{(2)})) \\ + i(\partial - \bar{\partial})(e^{2v-\phi}) \wedge (J_{(1)} + J_{(2)}) + i(\partial - \bar{\partial})(e^{2v-\phi+2C}) \wedge \pi^*J_{M^4}. \quad (82)$$

The  $F_3$  contains a nonzero piece  $ie^{-\phi}(\partial - \bar{\partial})J$ , hence the non-Kählerity of the eight-manifold  $M^8$  and the non-closure of  $J$  is closely related to the  $F_3$ . Acting on  $F_3$  by a further  $d$ ,

$$dF_3 = -dd^c(e^{-\phi}J) \\ = e^{2v-\phi}(\bar{F}_{(1)} \wedge F_{(1)} + \bar{F}_{(2)} \wedge F_{(2)}) \\ + d(e^{2v-\phi}) \wedge (i(\partial - \bar{\partial})(J_{(1)} + J_{(2)})) \\ - 2i\partial\bar{\partial}(e^{2v-\phi}) \wedge (J_{(1)} + J_{(2)}) - 2i\partial\bar{\partial}(e^{2v-\phi+2C}) \wedge \pi^*J_{M^4}. \quad (83)$$

The  $F_7$  is

$$F_7 = \frac{1}{2}e^{2A}\text{Vol}_{R^{1,1}} \wedge e^{-2A}d(e^{2A-\phi}J^2) = \frac{1}{2}\text{Vol}_{R^{1,1}} \wedge d(e^{2A-\phi}J^2). \quad (84)$$

According to Eq. (18) in Sec. 2,

$$\begin{aligned} d(e^{2A-\phi}J^2) &= -e^{4v+2A-\phi}(i\operatorname{Im}(F_{(1)}\wedge\bar{\theta}_{(1)})\wedge\theta_{(2)}\wedge\bar{\theta}_{(2)}+i\operatorname{Im}(F_{(2)}\wedge\bar{\theta}_{(2)})\wedge\theta_{(1)}\wedge\bar{\theta}_{(1)}) \\ &\quad +e^{2A-\phi}(d(4v+2A-\phi)\wedge J^2+4e^{4v+2C}dC\wedge\pi^*J_{M^4}\wedge(J_{(1)}+J_{(2)})). \end{aligned} \quad (85)$$

Using the relation (79), this can be simplified to

$$\begin{aligned} d(e^{2A-\phi}J^2) &= -e^{4v+2A-\phi}(i\operatorname{Im}(F_{(1)}\wedge\bar{\theta}_{(1)})\wedge\theta_{(2)}\wedge\bar{\theta}_{(2)}+i\operatorname{Im}(F_{(2)}\wedge\bar{\theta}_{(2)})\wedge\theta_{(1)}\wedge\bar{\theta}_{(1)}) \\ &\quad -4e^{4v+2A-\phi}dC\wedge J_{(1)}\wedge J_{(2)}. \end{aligned} \quad (86)$$

We use an identity

$$*_8((e^{2v+2C}F_{(i)})\wedge(e^v\operatorname{Re}\theta_{(1)})\wedge(e^{2v}\frac{i}{2}\theta_{(2)}\wedge\bar{\theta}_{(2)}))=-(e^{2v+2C}F_{(i)})\wedge(e^v\operatorname{Im}\theta_{(1)}), \quad (87)$$

in which the anti-self-duality (31) has been used. After using Eq. (87), we see that the pieces in (82) and (86) involving  $F_{(i)}$  are satisfied for the Eq. (76). Let us now look at the pieces in (82) and (86) which do not involve  $F_{(i)}$ . By comparing these pieces in Eq. (76), we see that

$$(\partial-\bar{\partial})(e^{2v-\phi})=0, \quad (88)$$

$$i(\partial-\bar{\partial})(e^{2v-\phi+2C})\wedge\pi^*J_{M^4}=*_8(2e^{4v-\phi}dC\wedge J_{(1)}\wedge J_{(2)}). \quad (89)$$

Hence, from the first equation above we have that

$$e^{2v}=e^\phi. \quad (90)$$

The second equation becomes

$$i(\partial-\bar{\partial})(e^{2C})\wedge(e^{2v+2C}\pi^*J_{M^4})=*_8(e^{4v}d(e^{2C})\wedge J_{(1)}\wedge J_{(2)}). \quad (91)$$

Using the metric (4) and a similar identity as (87),

$$d^c(e^{2C})\wedge J_{M^4}=*_4d(e^{2C}). \quad (92)$$

Acting on both sides by  $d$ ,

$$2i\partial\bar{\partial}(e^{2C})\wedge J_{M^4}=\frac{1}{2}\Delta(e^{2C})J_{M^4}\wedge J_{M^4}, \quad (93)$$

where  $\Delta$  is the Laplacian. Hence we have that

$$\begin{aligned} dF_3 &= e^{2v-\phi}(\bar{F}_{(1)}\wedge F_{(1)}+\bar{F}_{(2)}\wedge F_{(2)})-2i\partial\bar{\partial}(e^{2v-\phi+2C})\wedge\pi^*J_{M^4} \\ &= \bar{F}_{(1)}\wedge F_{(1)}+\bar{F}_{(2)}\wedge F_{(2)}-\frac{1}{2}\Delta(e^{2C})\pi^*J_{M^4}^2, \end{aligned} \quad (94)$$

where we have used the relation (90). Using the relations (79), (90) and the condition (80), we have that

$$e^{2C} = e^{-2\phi}. \quad (95)$$

Because of the anti-self-duality of the curvatures  $f_{(1)}$  and  $f_{(2)}$ ,

$$\bar{f}_{(i)} \wedge f_{(i)} = -\bar{f}_{(i)} \wedge *_4 f_{(i)} = -|f_{(i)}|^2 \text{Vol}_4 = -\frac{1}{2}|f_{(i)}|^2 J_{M^4}^2, \quad (96)$$

for  $i = 1, 2$ , where  $\text{Vol}_4$  is the volume form of  $M^4$  and we used the anti-self-duality  $f_{(i)} = -*_4 f_{(i)}^4$ . Hence,

$$\bar{F}_{(i)} \wedge F_{(i)} = -\frac{1}{2}|f_{(i)}|^2 \pi^* J_{M^4}^2. \quad (97)$$

The Bianchi identity for the  $F_3$  flux, in the presence of D5 and O5 sources, is

$$dF_3 = \rho^{(4)}(D5) - \rho^{(4)}(O5), \quad (98)$$

where  $\rho^{(4)}(D5)$  and  $\rho^{(4)}(O5)$  are the four-form Poincaré duals to the four-cycles that D5 and O5 wraps inside of  $M^8$ . The D5 is positively charged and the O5 is negatively charged. They are the source terms for Eq. (98). In the case that  $e^{2v-\phi}$  is constant,

$$dF_3 = -\frac{1}{2}|f_{(1)}|^2 \pi^* J_{M^4}^2 - \frac{1}{2}|f_{(2)}|^2 \pi^* J_{M^4}^2 - \frac{1}{2}\Delta(e^{2C})\pi^* J_{M^4}^2. \quad (99)$$

Hence, to balance the right hands of (98) and (99), we have

$$\bar{F}_{(1)} \wedge F_{(1)} + \bar{F}_{(2)} \wedge F_{(2)} - \frac{1}{2}\Delta(e^{-2\phi})\pi^* J_{M^4}^2 = \rho^{(4)}(D5) - \rho^{(4)}(O5). \quad (100)$$

This is a tadpole cancellation condition. On the right hand of Eq. (100), for general configurations, we have considered the inclusion of the negatively charged O5. The existence of negatively charged O5 has been anticipated in [32, 33, 34].

This is a configuration with  $F_3$  and  $F_7$  fluxes and dilaton, in the warped product (62) of two-dimensional Minkowski spacetime and  $SU(4)$ -structure eight-manifold  $M^8$  with the metric (4). The geometric model constructed in Sec. 2 is hence realized in the type IIB string theory.

## 5 Generalized calibrated cycles

In the previous sections, we have discussed the geometric model of the eight-manifolds and their realizations in type II string theory. Now we discuss more about specific geometric structures on these manifolds. A natural set of geometric structures are calibrated cycles, and in particular the generalized calibrated cycles in the presence of fluxes.

In the type II string theory, branes can wrap calibrated cycles. These cycles are calibrated by calibration forms. A usual calibration form is a closed form, and when restricted to the calibrated cycle, is the volume form [39]. There exist generalized calibration forms in the presence of background fluxes. These generalized calibration forms are not closed, due to the fluxes. However, the generalized calibration forms twisted by background form-potentials are closed. Meanwhile, the brane actions contain two types of terms. One type is the pull-back of volume, and another type in the brane action is the pull-back of background form-potentials, for example the R-R potentials. The generalized calibrations after subtracting the pull-back of background form-potentials, are hence closed. For more discussions on generalized calibrations, see for example [40, 41, 36] and references therein.

In the geometric model constructed in Sec. 2, we may denote the torus parametrized by  $x_1, y_1$  as  $T_{(1)}^2$ , and the torus parametrized by  $x_2, y_2$  as  $T_{(2)}^2$ . Consider a Kähler two-cycle  $\Sigma_{(1)}^2$  inside the base  $M^4$ , calibrated by  $J_{M^4}$ , so  $J_{M^4}|_{\Sigma_{(1)}^2} = \text{Vol}_{\Sigma_{(1)}^2}$ , which is the volume form of the two-cycle. We can make a four-cycle  $\Sigma_{(1)}^4$ , which is the restriction of the  $T_{(1)}^2$  fibration to the submanifold  $\Sigma_{(1)}^2 \subset M^4$ . Similarly, we can consider a Kähler two-cycle  $\Sigma_{(2)}^2 \subset M^4$ , calibrated by  $J_{M^4}$ , so that  $J_{M^4}|_{\Sigma_{(2)}^2} = \text{Vol}_{\Sigma_{(2)}^2}$ . In the similar way, we can make a four-cycle  $\Sigma_{(2)}^4$ , which is the restriction of the  $T_{(2)}^2$  fibration instead, to the submanifold  $\Sigma_{(2)}^2$  inside  $M^4$ .

Let us consider that the fivebrane is parallel to  $R^{1,1}$  and wraps a four-cycle  $\Sigma^4$  inside  $M^8$ . The worldvolume of the fivebrane is hence  $R^{1,1} \times \Sigma^4$ . In the case at hand, the brane action is  $S = -\mu_5 \int d^6\sigma e^{-\phi} \sqrt{-\det g_{\parallel}} \sqrt{\det g_{\perp}} + \mu_5 \int C_6$ , in which  $g_{\parallel}$  is the induced worldvolume metric on the  $R^{1,1}$  directions where  $\sqrt{-\det g_{\parallel}} = e^{2A}$ , and  $g_{\perp}$  is the induced worldvolume metric on  $\Sigma^4$ . The  $d^6\sigma$  is the volume element and the  $\mu_5$  is the charge of the brane. The brane configuration on the generalized calibrated cycle minimizes the total energy. This total energy is the sum of the energy coming from the tension on the worldvolume and that coming from the coupling of the brane to the background form-potential. The background form-potential here is  $C_6$ . We can write it as

$$C_6 = e^{2A-\phi} \text{Vol}_{R^{1,1}} \wedge \Pi_4. \quad (101)$$

The energy density of the fivebrane on  $\Sigma^4$  is  $E$ , and

$$\int d^4\sigma E = \int d^4\sigma e^{2A-\phi} \sqrt{\det g_{\perp}} - \int e^{2A-\phi} \Pi_4, \quad (102)$$

where  $d^4\sigma$  is the volume element of  $\Sigma^4$ . General discussions on calibrations on  $SU(4)$ -structure manifolds have been considered in [3]. The generalized calibration form  $e^{2A-\phi}\Xi_4$ , for any cycle  $\Sigma^4$ , satisfies the inequality

$$e^{2A-\phi}\Xi_4|_{\Sigma^4} \leq e^{2A-\phi} d^4\sigma \sqrt{\det g_{\perp}}|_{\Sigma^4} \quad (103)$$



where  $d^4\sigma\sqrt{\det g_\perp}|_{\Sigma'^4}$  is the volume form of  $\Sigma'^4$ , and the equality is satisfied for calibrated cycles. From the Eq. (84), the  $F_7$  is

$$F_7 = \text{Vol}_{R^{1,1}} \wedge d(e^{2A-\phi} \frac{1}{2} J \wedge J). \quad (104)$$

We also see from the Eq. (78) that

$$d(e^{2A-\phi} \text{Re}(e^{i\beta}\Omega)) = 0, \quad (105)$$

where  $e^{i\beta}$  is a phase factor. Since  $F_7 - dC_6 = 0$ , we have that

$$d[\text{Vol}_{R^{1,1}} \wedge e^{2A-\phi} \Xi_4 - C_6] = 0, \quad (106)$$

where

$$\Xi_4 = \frac{1}{2} J \wedge J + \text{Re}(e^{i\beta}\Omega). \quad (107)$$

This agrees with the observations in [3]. This means that  $e^{2A-\phi} \text{Vol}_{R^{1,1}} \wedge \Xi_4$ , after subtracting  $C_6$ , is closed. Hence, we can identify the generalized calibration form in the ten dimensions as

$$\begin{aligned} \Xi_6 &= \text{Vol}_{R^{1,1}} \wedge e^{2A-\phi} \Xi_4 \\ &= e^{2A-\phi} \text{Vol}_{R^{1,1}} \wedge \left( \frac{1}{2} J \wedge J + \text{Re}(e^{i\beta}\Omega) \right). \end{aligned} \quad (108)$$

The generalized calibration form twisted by the background form-potential is

$$\Xi'_6 = \Xi_6 - C_6. \quad (109)$$

According to Eq. (106), the  $\Xi_6$  after subtracting the background form-potential, is closed, that is,  $d\Xi'_6 = 0$ .

Inside  $M^8$ , the above generalized calibration form corresponds to  $e^{2A-\phi} \Xi_4$ . This form after subtracting the background form-potential  $e^{2A-\phi} \Pi_4$  is

$$\Xi'_4 = e^{2A-\phi} \Xi_4 - e^{2A-\phi} \Pi_4, \quad (110)$$

and  $d\Xi'_4 = 0$ . The  $M^8$  here is not a usual Kähler Calabi-Yau four-fold. In the case of the Kähler Calabi-Yau four-fold, the calibrations have been considered in [42]. According to Eq. (103), the restriction of the  $\Xi_4$  to a calibrated cycle is the volume form of the calibrated cycle. The restriction of the  $\Xi_4$  to the four-cycle  $\Sigma_T^4 = T^4$  is

$$\Xi_4|_{\Sigma_T^4} = -\frac{1}{4} e^{4v} \theta_{(1)} \wedge \bar{\theta}_{(1)} \wedge \theta_{(2)} \wedge \bar{\theta}_{(2)}|_{\Sigma_T^4}, \quad (111)$$

which is the volume form  $e^{4v} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$  of the cycle  $\Sigma_T^4$  inside  $M^8$ . This means that  $\Sigma_T^4$  is a generalized calibrated cycle. As constructed above, the four-cycle

$\Sigma_{(1)}^4$  is the restriction of the  $T_{(1)}^2$  fibration over the Kähler two-cycle  $\Sigma_{(1)}^2$  in  $M^4$ . The restriction of the  $\Xi_4$  to the four-cycle  $\Sigma_{(1)}^4$  is

$$\Xi_4|_{\Sigma_{(1)}^4} = e^{4v+2C} \left( \frac{i}{2} \theta_{(1)} \wedge \bar{\theta}_{(1)} \right) \wedge \pi^* J_{M^4}|_{\Sigma_{(1)}^4}, \quad (112)$$

which is the volume form of the cycle  $\Sigma_{(1)}^4$  inside  $M^8$ . Hence  $\Sigma_{(1)}^4$  is a generalized calibrated cycle. Similarly, the four-cycle  $\Sigma_{(2)}^4$  is the restriction of the  $T_{(2)}^2$  fibration over the Kähler two-cycle  $\Sigma_{(2)}^2$  in  $M^4$ . The restriction of the  $\Xi_4$  to the four-cycle  $\Sigma_{(2)}^4$  is

$$\Xi_4|_{\Sigma_{(2)}^4} = e^{4v+2C} \left( \frac{i}{2} \theta_{(2)} \wedge \bar{\theta}_{(2)} \right) \wedge \pi^* J_{M^4}|_{\Sigma_{(2)}^4}, \quad (113)$$

which is the volume form of the cycle  $\Sigma_{(2)}^4$  inside  $M^8$ . This shows that  $\Sigma_{(2)}^4$  is a generalized calibrated cycle. Hence, we have showed that the above generalized calibration form, after subtraction of the background form-potential, is closed, and that when restricted to the generalized calibrated cycle, is the volume form.

## 6 Discussion

The geometric model of  $T^4$  fibrations over Calabi-Yau two-folds constructed in this paper provides examples of eight-dimensional balanced manifolds and non-Kähler Hermitian manifolds. A seven-dimensional  $G_2$  manifold with torsion also occurs in the construction of the present paper. The eight-manifold of this type can also be viewed as a circle bundle over a  $G_2$  manifold with skew torsion. The eight-manifolds constructed here are used in ten-dimensional models in type IIB string theory. These models have rich geometric structures, such as fluxes and generalized calibrated cycles.

The IIB configurations here have similarities with configurations in heterotic string theory. The  $F_3$  flux in the type IIB case plays similar role as the  $H_3$  flux in the heterotic theory. The anomaly cancellation condition in the heterotic case can be viewed as a counterpart to the tadpole condition for fivebranes in the type IIB case here. The Hermitian Yang-Mills equations [26, 43] in the heterotic case, are counterparts to the generalized calibrations in the type IIB case at hand.

The  $T^4$  fibrations over Calabi-Yau two-folds considered here can also be used as background manifolds for heterotic string theory. In the case for heterotic theory, we need also to consider vector-bundles on these eight-manifolds. One can construct stable vector bundles over the eight-manifolds by pulling back stable bundles over the  $CY_2$  base space. Various methods of constructing vector-bundles in heterotic theory for six-manifolds may be used for eight-manifolds.

The non-Kähler geometries considered here would be useful for mirror symmetry for eight-dimensional non-Kähler manifolds [28]. The  $T^4$  fibration is analogous to the

$T^3$  in SYZ proposal [44], but for non-Kähler backgrounds. It may be interesting to perform T-duality transformations along  $T^4$ . The examples in this paper may serve as useful examples for performing T-dualities [45, 46] along higher dimensional tori.

It would be interesting to add  $H_3$  flux in the IIB case here and obtain more general configurations. In the presence of the  $H_3$  flux, the Dolbeault operators become twisted Dolbeault operators, which are twisted by  $H_3$  [3, 4, 33]. We leave these interesting and more general cases for future investigations.

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